A PROCEDURE FOR THE CALCULATION OF BOUNDARY LAYERS CONFINED IN DUCTS*

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It is shown that confined boundary layers can be calculated by a marching-integration procedure, if the unknown pressure gradient is determined from a formula valid for one-dimensional flow, and account is taken, in subsequent steps, of errors in the pressure gradients employed in earlier steps. The errors can be held down to any desired magnitude. The method is illustrated by reference to turbulent pipe flow and to the plane diffuser; calculations are provided of friction coefficients, Stanton numbers, and velocity and temperature profiles. Some comparisons with experiment are provided.

1. Introduction

1.1 The Problem. Boundary-layer theory is best developed for flows in which the variation of fluid pressure with longitudinal distance is prescribed; for example, Patankar and Spalding [1, 2] have provided a general procedure for solving the parabolic differential equations of momentum, enthalpy, and concentration in these circumstances. In many practical cases, however, the flow is confined in a duct, and the pressure gradient has to be calculated simultaneously with the other variables; this necessity leads to difficulty in applying the Patankar–Spalding procedure as originally formulated. It is the purpose of the present paper to show how the difficulty may be removed.

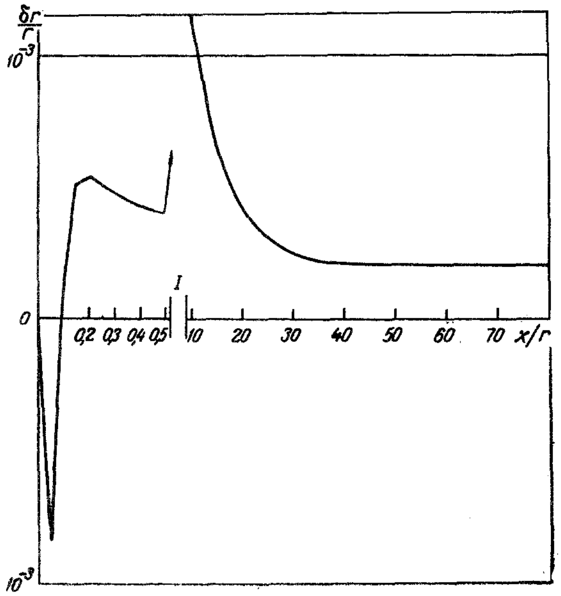

Fig. 1. The discrepancy between the thickness of the boundary layer and the radius of the pipe for a computation of flow in a smooth-walled pipe at a Reynolds number of 10^4. I) change of scale of diagram, and also change of forward-step size from 0.05 to 1.0 pipe radii.

1.2 The Method. Parabolic differential equations are most economically solved by a "marching" integration: values of velocity, temperature, etc., at one section across the boundary layer are deduced from the values of these variables that prevailed at corresponding locations a short distance upstream; and so, by a step-by-step movement in the downstream direction, the whole field of variables is determined. However, the calculation of the velocity profile at the downstream end of a step, from the velocity profile at the upstream end, requires knowledge of the pressure differential; and this is not available.

The procedure which is now recommended is:

(i) Guess the value of the pressure differential, taking into account the enlargement in the area of the duct, and the upstream velocity and density distributions.

*Paper contributed to the special issue of Journal of Engineering Physics devoted to the 60th birthday of Academician A. V. Luikov. The publisher thanks the author for providing the original manuscript.

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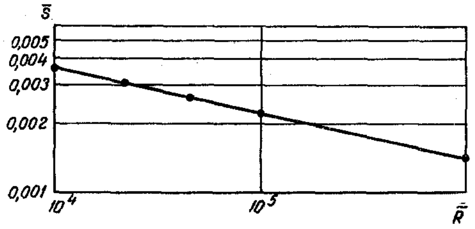

Fig. 2. Comparison of present predictions (•) with curve representing experimental data for the shear-stress coefficient versus the Reynolds number for fully-developed turbulent flow in a round-sectioned smooth-walled pipe.

(ii) Calculate the resulting downstream profiles of velocity, etc. In general, these will correspond to a boundary layer which does not quite fit the duct; there will be a small discrepancy between the flow area and the duct area.

(iii) Instead of modifying the pressure difference and repeating the integration for the same step, adjust the pressure difference to be applied to the next step so as to take account of the discrepancy between the flow area and the duct area that has just been observed.

There are many possible variants of this practice. One of the simplest is to calculate the pressure differential from the equations of conservation and momentum for a one-dimensional flow in a duct having the same average velocity and density as the actual flow; then we can deduce:

$$p_d - p_u = -P_u \frac{(x_d - x_u)}{A_u} \tau_u + \frac{\dot{m}\bar{u}_u}{A_u^2} (A_d - A_u), \tag{1}$$

where d and u are subscripts denoting downstream and upstream conditions respectively; p stands for pressure; P represents the periphery of the duct; x is the longitudinal distance; A stands for the cross-sectional area of the duct or flow; τ is the average shear stress at the wall of the duct; $\dot{m}$ is the rate of mass flow along the duct; and $\bar{u}$ stands for its average velocity.

The most important part of the procedure is the correct evaluation of $(A_d - A_u)$: the duct area must be taken for the downstream station, and the flow area for the upstream station. Sometimes, in order to avoid instability in the forward-integration process, it is helpful to insert only a fraction (say one half) of this discrepancy; but the procedure is a very tolerant one, the differences between practices resulting merely in different magnitudes for the average area discrepancy along the length of the duct.

The workings of the method will now be illustrated by two examples: fully-developed turbulent pipe flow; and the plane diffuser.

2. Turbulent Pipe Flow

2.1. Physical Inputs to the Calculation. The process of turbulent flow in a pipe is too well known to require description. The transport processes within it can be presumed to obey the mixing-length hypothesis of Prandtl [3], with the mixing-length distribution as specified by Nikuradse [4, 5], with values increased by the factor 0.41/0.4; the turbulent Prandtl number will be taken as 0.9. Near the wall, laminar effects become important; these can be accounted for by supposing that, across a thin region which contains the semi-laminar layer, the shear stress coefficient s is related to the Reynolds number R of the region by:

$$s = [K/\ln(1 + ERs^{1/2})]^2, \tag{2}$$

where K = 0.41 and E = 7.35; and the resistance to heat and mass transfer is supposed to obey the formula recommended by Spalding and Jayatillaka [6].

These specifications suffice to determine the basic transport properties of the turbulent pipe flow; the task of the numerical integration procedure is then to determine the consequent profiles of velocity, temperature, and concentration, and the associated relations between pressure-drop coefficient, Stanton number, Prandtl number, and Reynolds number of the pipe. Of course, at the same time, it is possible (indeed necessary) to calculate the stages by which the fully-developed profiles are attained with increasing distance from the entrance; but these will not be reported here because attention will be focussed on the flow-development problem in part 3 of this paper.

2.2. Mathematical Inputs. The Patankar–Spalding procedure is used for solving the partial differential equations. The grid has 30 intervals in the radial direction, distributed so that the stream-function differences are approximately equal. The forward-step size is equal to one pipe radius, except that the first

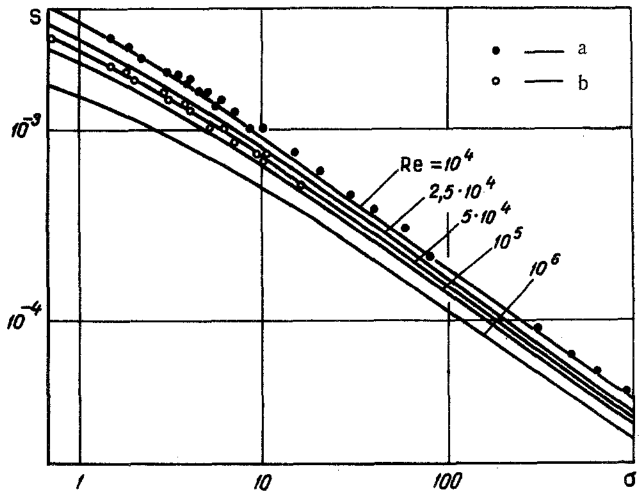

Fig. 3. Comparison of predicted values (curves) and experimental values (points) for the Stanton number in fully developed turbulent flow in a smooth-walled pipe at various laminar Prandtl numbers and Reynolds numbers. The points, taken from a survey by Deissler [8], are for Reynolds numbers of 10^4 (a) and $5 \cdot 10^4$ (b).

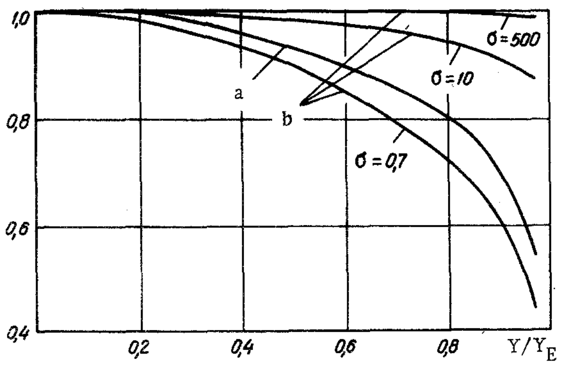

Fig. 4. Profiles of velocity (a) and of conserved property ϕ (b), across a fully-developed turbulent pipe flow at a Reynolds number of 10^4. $\sigma \equiv$ laminar Prandtl–Schmidt number.

twenty steps are of one-twentieth this size. The area difference $(A_d - A_u)$ was multiplied by 0.5 in order to ensure stability under all circumstances.

2.3. Results. The Extent to which the Flow Fits the Duct. Fig. 1 shows how successful is the practice outlined in section 1.2 above; the abscissa is the longitudinal distance x/r, where r is the pipe radius; and the ordinate is δ_r/r, where δ_r is the excess of the radius occupied by the calculated flow over the actual pipe radius r. Evidently the discrepancy is a very small percentage, especially in the fully-developed condition; it can be made still smaller whenever desired, simply by reduction of forward-step size, or by other measures.

Physical Predictions for Fully-Developed Flow. Figs. 2 and 3 contain the results of predictions carried out at various Reynolds numbers. In the first diagram, the predictions are represented by the points; they lie perfectly on the curve which represents the experimental data as correlated by Schlichting [7]. In the second, which has Stanton and Prandtl–Schmidt numbers as ordinate and abscissa, the curves represent the results of the computations, while the experimental data, extracted from a survey by Deissler [8], are represented by points.

Figure 4 displays a few of the many profiles of velocity and ϕ (the conserved property, either enthalpy or concentration) which were computed during the investigation.

Discussion. The results shown in these four figures are in satisfactory agreement with experimental data, where checks have been made. This is not surprising, because the physical inputs (i.e., the Nikuradse mixing-length distribution, the Spalding–Jayatillaka formula, etc.) are correlations of data obtained in this very pipe-flow situation. The agreement must therefore merely be regarded as a demonstration of the fact that the proposed computation procedure is capable of application, and introduces no error of its own.

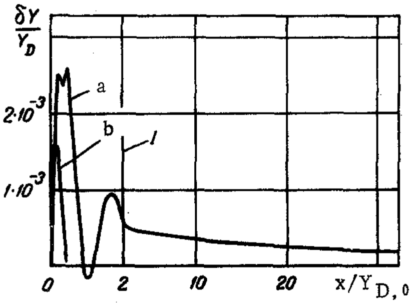

Fig. 5

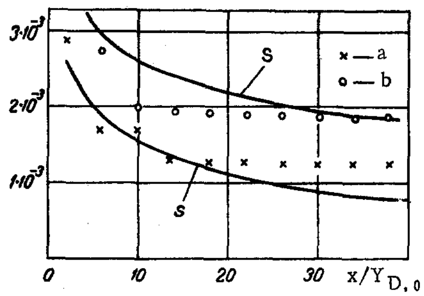

Fig. 6

Fig. 5. Variation with longitudinal distance of the discrepancy between the duct width and the calculated flow width, for the 4° diffuser. I) Change of horizontal scale. a) $\delta x/Y_{D,0} = 0.1$; b) 0.05.

Fig. 6. Variation of drag coefficient s and Stanton number S along the wall of the 4° diffuser, compared with experimental data (a for s, b for S) of Hool [9].

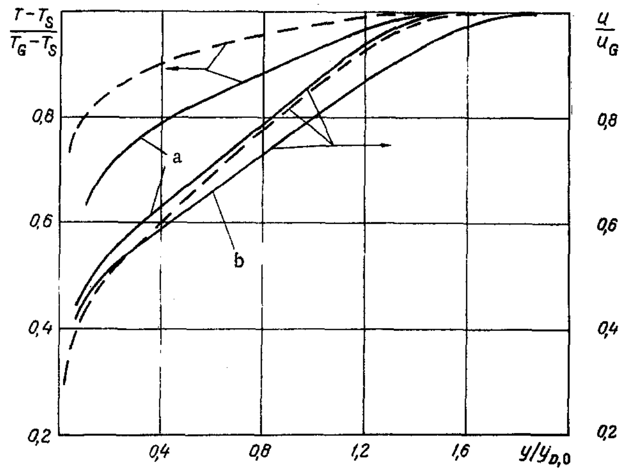

Fig. 7. Velocity and temperature profiles in the 4° diffuser, 34 initial half-widths from the entrance to the duct. Full lines represent the results of computation; broken lines represent the experimental data of Hool [9]. The Reynolds number $u_{G,0}Y_{D,0}/\nu$ equals 10^5 for experiment and computation alike.

3. The Plane Diffuser

3.1. The Process Considered. There are surprisingly few full reports in the literature of measurements of friction and heat transfer in confined-flow situations other than ducts of uniform area. One of the few is that of Hool [9], who published data for a plane-walled diffuser, having one steam-heated side, and enclosing a steady flow of air; it is with this configuration that the present section will be concerned. Unfortunately, there is no information in the report about the profiles of temperature and velocity that prevailed at the entry to the measuring section; these have therefore had to be guessed. Both profiles have been presumed to have initially the seventh-power-law form, and to have the same thickness; various guesses have been made for the value of this thickness.

3.2 Physical Inputs. The mixing-length hypothesis has been used once again; this time, the mixing length has been taken as 0.435 times the distance from the wall, up to a maximum of 0.09 times the boundary-layer thickness. The region near the wall was handled by means of the "wall functions" described in detail by Patankar and Spalding [2]. The laminar Prandtl number was taken as 0.7, and the turbulent one as 0.9.

Computations were made for two angles of divergence: 4° and 7°.

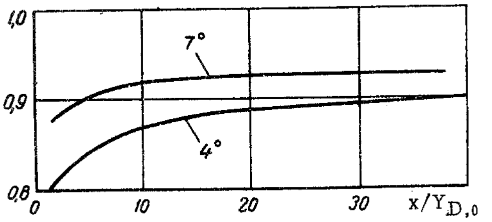

Fig. 8. Variation with longitudinal distance of the efficiencies of 4° and 7° plane diffusers.

3.3. Mathematical Inputs. Once again there were 30 grid intervals in the cross-stream direction; but these did not, as in the case of pipe flow, stretch all the way across the duct: they were confined to the region of significant shear stress, i.e., to the boundary layer proper, in the way that is characteristic of the Patankar–Spalding method, and which gives it its especial economy. The forward-step size was chosen as a fixed quantity for the whole of an integration; various values were used, to be mentioned below.

The pressure difference for a forward step was calculated, for variety, in a way that differed slightly from that described above: the pressure gradient actually used was equal to 0.2 times the value given by Eq. (1) plus 0.8 times the value that prevailed at the previous station. This "historical weighting" is another way of nullifying any tendency to instability that should ensue from our reluctance to introduce iterative operation.

3.4. Results. The Extent to which the Flow Fits the Duct. Figure 5 shows the variation with downstream distance x of the discrepancy δY between the area occupied by the calculated flow, and that provided by the diffuser Y_D; once again, it is seen that the value is agreeably small; and it is also evident that this value diminishes when smaller values are taken for the forward-step size δx ($Y_{D,0}$, by which δx is divided, is the value of Y_D at the diffuser entrance). There are no significant effects of step size changes on such physically-meaningful quantities as Stanton number.

Stanton Number and Drag Coefficient. Figure 6 shows curves for the predicted values of shear-stress coefficient s and Stanton number S for the 4° diffuser. The initial boundary-layer thickness was taken as one-twentieth of $Y_{D,0}$. Experimental values reported by Hool [9] are represented by circles and crosses.

Evidently the predicted and experimental values are of the same order. There is a tendency for the shear-stress predictions to lie below the measurements, while the Stanton-number predictions show the opposite behavior. While incorrect inputs or presumed starting profiles may be in part to blame, it should also be mentioned that the reported constancy of s and S in the downstream region is not very plausible. There is need for a more complete investigation, both experimental and theoretical, in order to resolve these questions.

Profiles. Figure 7 displays some velocity and temperature profiles for a station near the outlet of the 4° diffuser. The full lines represent the predictions; the broken ones the reported experimental findings. Because the thickness of the boundary layer at the entrance to Hool's diffuser is not known, velocity profiles are presented which result from two distinct integrations: with this thickness equal to 0.05 $Y_{D,0}$ in one case and 0.20 $Y_{D,0}$ in the second. Inspection of the diagram shows that the first starting point yields a velocity profile which agrees quite well with the experimental one, both in shape and thickness.

The success in predicting the velocity profile contrasts with the failure, revealed by the top two curves of Fig. 7, to compute a realistic temperature profile. This cannot be attributed to an incorrect choice of Prandtl number, for no value in the plausible range (0.5 to 1.0) would significantly improve the agreement; once again, the explanation of the discrepancy must be sought from a more complete experimental and theoretical study.

Diffuser Efficiency. Of course, computations of the kind which yield friction and heat transfer provide very much additional information. To illustrate this point, Fig. 8 displays the values of the predicted efficiencies of the 4° and 7° diffusers, defined as actual pressure rise divided by the pressure rise that would result from one-dimensional isentropic flow. No experimental data for this quantity were reported by Hool; but the predicted values are plausible in magnitude and trend.

CONCLUSIONS

(a) There now exists a numerical procedure which is easily capable of solving, exactly and numerically, the partial differential equations of steady boundary-layer phenomena, even when the flows are confined in ducts.

(b) Knowledge of the basic transport processes in turbulent flows suffices completely for the accurate prediction of heat, mass, and momentum transfer in turbulent pipe flow and partially for that in a plane diffuser. In the latter case however, there are some discrepancies between the predictions and the experimental data, which only a new investigation can resolve.

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